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LETTER TO THE EDITOR

Parisi's mean-field solution for spin glasses as an analytic continuation in the replica number

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Abstract. Parisi's replica symmetry breaking solution for spin glasses is extended to finite replica number n. The free energy $F_{\mathrm{P}}(n)$ obtained this way, as well as its first two derivatives with respect to n, are shown to join the corresponding values in the Sherrington–Kirkpatrick (SK) solution at a characteristic value $n_{\mathrm{s}}(T)$, where stability breaks down in the latter. The continuation composed of the SK branch $F_{\mathrm{SK}}(n)$ for $n \geqslant n_{\mathrm{s}}(T)$ and the Parisi branch $F_{\mathrm{P}}(n)$ for $0 \leqslant n \leqslant n_{\mathrm{s}}(T)$ fulfils the requirements of convexity, monotonicity and stability for all n.

The replica method (Hardy *et al* 1934, Kac 1968, Lin 1970, Edwards 1972, Edwards and Anderson 1975, Grinstein and Luther 1976, Emery 1975) seeks to evaluate the average of a logarithm through the innocent-looking formula

$$\langle \ln Z \rangle = \lim_{n=0} \frac{1}{n} \ln\langle Z^n \rangle. \tag{1}$$

In typical applications Z is the partition function of a system with random interactions, the average is over the randomness and the left-hand side is basically the quenched free energy, $F = -T\langle \ln Z \rangle$. The meaning of

$$\psi(n) = (1/n) \ln\langle Z^n \rangle \tag{2}$$

on the right is obvious as long as n, the number of replicas, is a positive integer. In order to take the limit $n = 0$, however, one has to continue ψ from integer to real n, which is, in general, a largely underdetermined task. Indeed, in a critical analysis of the possible pitfalls of the replica method as applied to the Sherrington–Kirkpatrick (1975) problem, van Hemmen and Palmer (1979) came to the conclusion that there had to exist at least two distinct continuations in that problem: the 'obvious' continuation, consisting in interpreting n simply as a real variable in formulae derived on the integers, which was used by SK and led them to unacceptable low-temperature results, and another, unknown continuation, which was to be expected to produce sensible results.

The fact that $\psi(n)$ is related to the nth moment of a random variable imposes some restrictions on the possible continuations. As pointed out by van Hemmen and Palmer, (2) implies e.g. that $n\psi(n)$ must be a convex function of n. Derrida (1980), in the context of his random energy model, observed that this property breaks down at low temperatures at some characteristic $n = n_{\mathrm{c}}(T)$ (c stands for convexity) below

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which the 'obvious' continuation becomes meaningless; he also proposed a simple Maxwell-construction-like extrapolation from n_c down to $n=0$, which turned out to reproduce the correct low-temperature results, known exactly for this model (Derrida, private communication). Rammal (1981) followed a similar approach in the SK model. To the condition of convexity he added the further requirement that $\psi(n)$ must be a non-decreasing function of n. The proof of this property can be obtained by substituting $X=Z^n$, $Y=1$ and $r=n'/n$, $n'>n$, into the Hölder inequality

$$\langle XY\rangle \leqslant \langle X^r\rangle^{1/r}\langle Y^s\rangle^{1/s}, \qquad r>1, \qquad 1/r+1/s=1,$$

valid for any positive random variables X, Y. Rammal also observed that both convexity and monotonicity broke down in the SK solution below T_c, at some characteristic values $n_c(T)$ and $n_m(T)$, respectively, with $n_m(T)$ being apparently the larger of the two for all $T\leqslant T_c$. Therefore $n_m(T)$ can be considered as a lower bound to the range of n values where the SK solution can be accepted. This lower bound is rather small for all T: near $T_c=1$ it vanishes with the reduced temperature $\tau=(T_c-T)/T_c$ as $n_c(T)=\tau+\ldots$, it reaches a maximum of about 0.2 around $T\sim 0.5\,T_c$ (Rammal 1981), while at low T it goes like $n_c(T)=(2\ln(4/3))^{1/2}T+\ldots$. In view of the smallness of $n_m(T)$, it seemed reasonable to try an extrapolation from n_m down to $n=0$: the one proposed by Rammal consisted in replacing the SK free energy $F(n)$ by its minimum $F(n_m)$ for all $0<n<n_m$. The results for various physical quantities obtained in this way turned out to be in fair agreement, though certainly not identical, with the predictions of Parisi's (1979, 1980) replica symmetry breaking solution.

In addition to the requirements of convexity and monotonicity in n, there is an all-important third condition that any meaningful continuation has to fulfil, namely the condition of stability. This means that the matrix of second derivatives of F with respect to the components of the order parameter $q_{\alpha\beta}$ must have non-negative eigenvalues. As is well known from the work of de Almeida and Thouless (1978), the SK solution is unstable at $n=0$. Since, on the other hand, it is evidently stable for positive integer n, it is natural to introduce a third characteristic n value, to be called $n_s(T)$, associated with the breakdown of stability. The dangerous eigenvalue of de Almeida and Thouless (AT) can be written as

$$\lambda(n)=1-T^{-2}[\cosh^{-4}\xi]_n \tag{3}$$

where $[\ldots]_n$ is an 'average' with respect to the weight function

$$\frac{\exp(-z^2/2)\cosh^n\xi}{\int_{-\infty}^{\infty}\mathrm{d}z\,\exp(-z^2/2)\cosh^n\xi} \qquad \text{with } \xi=q^{1/2}z/T \tag{4}$$

and q is determined by $q=[\tanh^2\xi]_n$. The definition of n_s is then $\lambda(n_s)=0$ (if there are more roots, the largest is meant).

At this point one is naturally led to inquire about the relationship between $n_s(T)$ and the previous characteristic n's. The first observation we make in this paper is that the largest, hence the only relevant, of the three is n_s. Indeed, from (3), (4) it is easily seen that $n_s=\frac{4}{3}\tau+\ldots$ near T_c, while $n_s=T(2\ln(1/T))^{1/2}$ at low temperatures; therefore it is larger than $n_m(T)$ which in turn is larger than $n_c(T)$ at both extremes, and it seems probable that this holds true also for all intermediate T values. This means that the breakdown of stability prevents one from reaching Rammal's lower bound and, at the same time, raises the question whether it is possible to find a natural continuation of the SK branch starting from n_s and fulfilling all the requirements discussed above for $0 \leqslant n \leqslant n_s$. The second point of this paper is that, at least near T_c, this is quite possible: the continuation found possesses all the required properties, joins the SK branch at n_s smoothly (the free energy and its first two derivatives are continuous at n_s) and, most remarkably, goes over into Parisi's solution at $n=0$. The rest of the paper is a demonstration of this statement.

Close to T_c the free energy functional (or $\psi(n)$ itself) can be expanded in $q_{\alpha\beta}$. Dropping an irrelevant constant term, absorbing a factor T^{-2} into q and keeping only that one of the quartic terms which is responsible for the replica instability, one arrives at the truncated model expression (Parisi 1979)

$$\psi(n) = -\frac{1}{T}F(n) = \frac{1}{2n}\left(\tau \operatorname{Tr} q^2 + \tfrac{1}{3}\operatorname{Tr} q^3 + \tfrac{1}{6}\sum_{\alpha\neq\beta} q_{\alpha\beta}^4\right). \tag{5}$$

Because of the omitted q^4 terms (5) is not a consistent expansion in τ, hence there is some arbitrariness in the coefficient in front of the quartic term. The value we have chosen (following Parisi) is such that it reproduces all the quantities which are of importance for us (the eigenvalues of the Hessian, the values of the critical n's) correctly to leading order in τ. An additional remark is that, in the same spirit as Parisi, we regard (5) as a closed model expression and do not consider the effect of the neglected terms when making expansions in τ. In order to see the difference between the SK and Parisi solutions we have to work to $O(\tau^5)$, where we stop, for simplicity.

Let us first work out the replica symmetric (SK) solution for (5). Assuming $q_{\alpha\beta} = q(1-\delta_{\alpha\beta})$, we find the stationarity condition

$$2\tau + (n-2)q + \tfrac{2}{3}q^2 = 0. \tag{6}$$

Substituting the solution of (6) back into (5) and assuming n to be of the order of τ, we obtain

$$-\psi_{SK}(n) = \tfrac{1}{6}\tau^3 + \tfrac{1}{12}\tau^4 + \tfrac{1}{18}\tau^5[1 + 3(n/2\tau)(1 - n/2\tau)] + \dots. \tag{7}$$

The dangerous eigenvalue of the Hessian associated with (5) can easily be obtained by the method of AT to be

$$\lambda(n) = 2\tau - 2q + 2q^2 = q(n - \tfrac{4}{3}q). \tag{8}$$

On the basis of (6), (7) and (8) it is easy to check that the convexity of $n\psi(n)$ breaks down at $n_c = \tfrac{2}{3}\tau$, monotonicity of $\psi(n)$ at $n_m = \tau$ and stability at $n_s = \tfrac{4}{3}\tau$, the same values as we have found in the complete model. Because of the breakdown of stability the symmetric solution becomes meaningless below n_s.

In order to find an acceptable continuation into the range $0 < n < n_s$, we propose to consider a Parisi-type ansatz for the matrix $q_{\alpha\beta}$, i.e. to make a hierarchical subdivision of the matrix into blocks of size $p_1, p_2, \dots, p_R$, with matrix elements q_l, $l = 0, 1, 2, \dots, R$ on the lth level of hierarchy, substitute it into (5) and continue the resulting formula in n and the p_l's into the interval [0, 1]. Upon continuation the matrix $q_{\alpha\beta}$ is 'turned inside-out', the original order of n and the p_l's is reversed and becomes $n \equiv p_0 < p_1 < p_2 < \dots < p_R < p_{R+1} \equiv 1$. The only difference with respect to Parisi's scheme is that n will not be allowed to go to zero, but will be considered to be a small positive number. Next the continuous limit $R \to \infty$ is performed, whereupon the p's fill the interval $[n, 1]$ quasi-continuously, $p_l = n + (1-n)l/(R+1)$, and an order parameter function

$q(x)$, $n \leqslant x \leqslant 1$, is defined as by Parisi. Working out (5) for this ansatz yields

$$-\psi_{\mathrm{P}}(n)=\tfrac{1}{2}\tau\int_n^1 \mathrm{d}x\, q^2(x)-\tfrac{1}{6}\int_n^1 \mathrm{d}x\left(xq^3(x)+3q^2(x)\int_x^1 \mathrm{d}t\, q(t)\right)+\tfrac{1}{12}\int_n^1 \mathrm{d}x\, q^4(x). \tag{9}$$

This is stationary with respect to variations of q at

$$q(x)=\begin{cases} q_0, & n\leqslant x\leqslant x_0,\\ \tfrac{1}{2}x, & x_0\leqslant x\leqslant x_1,\\ q_1, & x_1\leqslant x\leqslant 1,\end{cases} \tag{10}$$

where $x_0=2q_0=\frac{3}{2}n$ and $x_1=2q_1$ is determined by the condition $\tau-q_1+q_1^2=0$, the same as in the original Parisi scheme corresponding to $n=0$. The stationary solution (10) is almost exactly the same as Parisi's $q(x)$ in a field; the only difference is that here $q(x)$ is not defined for $x<n$. The solution (10) is meaningful only as long as $q_0\leqslant q_1$, that is $n\leqslant n_{\mathrm{s}}(T)=\frac{4}{3}q_1(T)$. Comparison with (6) and (8) shows that the upper bound $n_{\mathrm{s}}(T)$ for the validity of (10) is the same as where the stability of the SK solution broke down. At this critical value of n a de Almeida–Thouless (1978) type of transition is taking place.

Substitution of (10) into (9) gives

$$-\psi_{\mathrm{P}}(n)=\tfrac{1}{6}\tau^3+\tfrac{1}{12}\tau^4+\tfrac{1}{10}\tau^5-\tfrac{2}{135}(\tfrac{3}{4})^5n^5+\ldots,\qquad 0\leqslant n\leqslant n_{\mathrm{s}}(T). \tag{11}$$

From (7) and (11) one easily sees that $\psi_{\mathrm{P}}(n)$ and $\psi_{\mathrm{SK}}(n)$, along with their first two derivatives, coincide at $n_{\mathrm{s}}(T)$: the 'transition' is of third order. The analogy with the AT transition makes me believe that this remains true also for the complete model. (Note that a Rammal-type extrapolation, starting from the minimum of $\psi_{\mathrm{SK}}(n)$, would produce a second-order transition: one cannot match the second derivative at an extremum while trying to save monotonicity.)

The Parisi branch (11) obviously fulfils the requirements of monotonicity and convexity. As for the stability, its demonstration takes a rather long calculation whose details are clearly impossible to describe here. The proof follows the stability analysis of the ($n=0$) Parisi solution (De Dominicis and Kondor 1983), or even more closely the same type of analysis in a field (Kondor and De Dominicis 1983), and arrives at the result that the eigenvalues of the Hessian form two bands: the band of large (order τ) eigenvalues of width $\sim\tau^2$ and that of the small ($\sim\tau^2$) eigenvalues, with lower edge zero for any $0<n<n_{\mathrm{s}}$. As n approaches $n_{\mathrm{s}}=0$ the bandwidths vanish and the spectrum goes over into the two (highly degenerate) AT eigenvalues 2τ and 0 corresponding to $n=n_{\mathrm{s}}$. The Parisi branch is therefore marginally stable for all $0\leqslant n\leqslant n_{\mathrm{s}}$.

I conclude with a few remarks. Though the observations above were made on the truncated model, they may nevertheless have a certain heuristic value also for the full SK problem. They emphasise again the role of stability considerations and may perhaps give additional credence to the Parisi solution by demonstrating how it can be placed in the mathematically minded approach initiated by van Hemmen and Palmer (1979). I have not been able to say anything about the uniqueness of the continuation found; the proof of that would probably be equivalent to the proof of global stability of the Parisi solution, undoubtedly a very hard problem. Finally, I point to a most amusing possible application of the above approach: the vanishing of $n_{\mathrm{s}}(T)$ at low temperatures, combined with the assumption of a third-order transition, invites one to attempt to guess the zero-temperature properties of the Parisi solution from an extrapolation starting from the SK solution near $n_s(T)$. This is left for a later work.

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